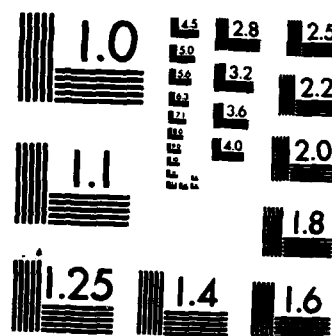AD-A139 616   KINETIC THEORY OF GASES MAGNETO-FLUID DYNAMICS AND   1/1
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This report summarizes progress to date under the grant. The areas covered in this report are (1) mathematical theory of queer differential equations (QDE); (2) universal solutions in multidimensional diffusion equations; (3) exact integrals of the Emden-Fowler equation; (4) new results in the theory of turbulent self-diffusion; and (5) mathematical theory of the essential spectrum in magnetohydrodynamics.

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AFOSR   Interim Technical Report
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January 20, 1984

The following summary is a progress report for the research
currently carried out under AFOSR Contract # AFOSR 81-0020.   The
areas covered in this report are 1)mathematical theory of queer
differential equations (QDE), 2) universal solutions in multi-
dimensional diffusion equations, 3) exact integrals of the Emden-
Fowler equation, 4) new results in the theory of turbulent self-
diffusion and 5) mathematical theory of the essential spectrum in
magnetohydrodynamics.

1.   The Theory of Queer Differential Equations (QDE's)

Since 1974, the use of QDE's in the theory of adiabatic and
diffusing plasma media has proved very useful.   More recently,
strong interest has been placed on extending the use of QDE's to
other problems of computational fluid interest where there are
multiple time scales such that they can be separated sufficiently
to allow quasi-static evolution of part of the solution.   Large
scale meteorological computations are envisioned as candidates
for such an application of QDE techniques.   Numerical algorithms
have been developed which accelerate large diffusion codes and the
convergence properties of the numerical schemes depend, to a large
extent, on as yet incomplete understanding of the theory of QDE's.
Recently, research in the mathematical nature of QDE's has been
pursued by H. Grad, P. Laurence and E. Stredulinsky.   We present
here a progress report of this work.

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Queer differential equations were introduced by Harold Grad to describe a new class of functional differential equations which model the slow adiabatic diffusion of a plasma through a magnetic field. The prototype for such an equation is

$$\Delta\psi = F(x,V,\psi,\psi',\psi'') \tag{1}$$

where ' denotes differentiation with respect to V the volume enclosed with level sets of $\psi(x)$. Here we recall ψ^* is the increasing rearrangement of a function. It is essentially the inverse function of the better known distribution function of real analysis, $V(t)$, where,

$$V(t) = \left| x: \psi(x) < t \right|,$$

($| \cdot |$ means Lebesgue measure)

Over the past year research has continued on the model queer differential equation,

$$\Delta\psi = -(\psi^*)'' \quad (V)$$

Equations of this type arise in the work of H. Grad in the theory of the adiabatic compression of a plasma.[1,2]

One approach to this problem is variational (See Proposal, May '83). We study the problem of minimizing, for bounded $\Omega \subset R^n$, $T(\psi)$, where

$$T(\psi) = \int_\Omega |\nabla\psi|^2 dx + \int_o^{|\Omega|} \psi^{*\,'2} dv \tag{1}$$

for $\psi = 0$ on $\delta\Omega$, $\psi^*(0)=0$, $\psi^* (|\Omega| = 1$ (2)

In other words $\psi(x)$ is zero at its infimum, and one at its supremum.
Results already announced in the last interim report are that $T(\psi)$
has a minimizer in a function class that incorporates the boundary
conditions (2).

Several regularity results have been established for this pro-
blem, the most important being that ψ^* is a Lipschitz continuous
function of V and that $\int\limits_{\{\psi=t\}} |\nabla\psi|$ is bounded below in terms of an expres-
sion that involves only the measure of the level set $\{\psi=t\}$ (see [3]).

It rapidly became apparent that in order to obtain initial re-
sults more quickly, at least initially, it would be necessary to
separate questions concerning the structure of the level sets and cri-
tical points of the minimizer ψ from its differentiability (of whatever
order). For this reason, an auxiliary variational problem is intro-
duced [3] for convex Ω that minimizes $T(\psi)$ among functions where
level sets are convex. There is strong evidence that a minimizer to
this auxiliary problem provides a solution to the queer differential
equation 1), even though the admissibility class of trial functions
has been narrowed. All details regarding this are not complete as of
this writing although many are contained in [4]. In any case this is
why aside from, we think, being interesting in their own right, esti-
mates for the auxiliary problem will carry over to the original problem.

For the auxiliary problem we have introduced an approximate prob-
lem by modeling the QDE term through a finite difference. The nth

approximation gives rise to a free boundary problem with n free boundaries which is a topic that is in and of itself on the frontiers of work in free boundaries (e.g., in the work of Luis Caffarelli and Avner Friedman). It can be shown that the approximate problem is harmonic in the regions contained between the free boundaries and that certain jump conditions on the gradient of the function are satisfied across the free boundaries. A combination of the use of these jump conditions with novel results on V(t) for harmonic functions permits one to conclude that the solution to our approximate QDE problem is superharmonic, which we feel is itself a significant achievement. Moreover, as the property of being superharmonic is preserved under weak H^1 limits as $n \uparrow \infty$, it is expected that the superharmonicity result will carry over to the full auxiliary problem and thus via the remarks above to the full QDE.

As we have exhibited a strong interplay between understanding certain free boundary problems for harmonic functions and queer differential equations, an interaction between us and L. Caffarelli and A. Friedman has been sparked which has already led to a few new theorems which will also be contained in [4]. Furthermore, current work of theirs [5] for the case where Ω is not convex provides a significant boost to the idea of attempting a similar approximation procedure by an n-shell free boundary problem in the nonconvex case. Queer differential equations give rise to many new and interesting questions about free boundary problems, thus we think they are a source for challenging problems for years to come in this area.

It was claimed by Harold Grad long ago that "plasma physics continually gives rise to new and interesting mathematical structures." The relationship we have sketched between queer differential equations and free boundary problems provides another example of the truth of Grad's conjecture.

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2. Universal Solutions in Multidimensional Diffusion Equations

This work is presently being carried out by P. Rosenau and represents a natural extension of his work in non-linear thermal and diffusive waves described in the Interim Report of January 31, 1983. Further, this work represents a major advance in the mathematical structure of coupled diffusion equations.

As a model for such a system of equations we consider the transport of mass and energy through coupled transport equations. The interaction between mass and heat diffusion (transport) coefficients is a source of many new effects described by the following prototype system of equations:

$$\partial_t \rho = \partial_x D_1(\rho, T) \partial_x \rho; \tag{1}$$

$$\rho \partial_t T = \partial_x \rho D_2(\rho, T) \partial_x T; \tag{2}$$

and

$$D_1 = D_{o1} \rho^{\alpha_1} T^{\beta_1}, \quad D_2 = D_{o2} \rho^{\alpha_2} T^{\beta_2}; \tag{3}$$

where ∂_t and ∂_x are partial derivative operators, α, β, D_{o1} and D_{o2} are constants and $x \in [-1,1]$. We assume slab symmetry, but our results are valid for other symmetries as well. ρ and T represent mass and temperature respectively.

The initial date for density $\rho(x,o)$ and temperature $T(x,o)$ are defined with homogeneous boundary conditions of either the first type (Dirichlet),

$$T(\pm 1, t) = 0, \quad \rho(\pm 1, t) = 0, \tag{4}$$

or the third type (convective),

$$\partial_x T \pm h_e T = 0 \ , \ \partial_x \rho \pm h_\rho \rho = 0 \text{ at } x = \pm 1, \tag{5}$$

where h_e and h_ρ are constants.

For Dirichlet boundary conditions [Eq. (4)], the elliptic parts of Eqs. (1) and (2) are degenerate on the boundary, and a solution may exist only in a weak sense. This singular behavior of the equations on the boundary prevents either density or temperature from vanishing when

$$\alpha_1 \leq \alpha_2 \text{ and } \beta_1 < \beta_2 . \tag{6}$$

When this occurs, no zero Dirichlet boundary conditions should be imposed on the temperature.

Similarly, when

$$\alpha_1 > \alpha_2 \text{ but } \beta_1 \geq \beta_2 + 1, \tag{7}$$

the density must be bounded but cannot be required to vanish. When $\alpha_1 = \alpha_2$ and $\beta_1 < \beta_2$, density blocks the diffusion of heat and the diffusing plasma becomes isothermalized. When $\beta_1 = \beta_2$ and $\alpha_1 > \alpha_2$, particles cannot escape and the system evolves toward a cold constant-density state.

To simplify our work initially we have concentrated on the convective boundary conditions [Eq. (5)].

From previous studies of a single diffusion equation describing diffusion of either mass or energy, we expect a simple pattern to emerge out of nearly arbitrary initial data. The

separable solutions of Eqs. (1) and (2) must satisfy special initial conditions, but they are of prime importance. Indeed, extensive numerical experimentation has shown for almost arbitrary initial conditions that the system, after a short transient time, either evolves toward a time-space separable solution or actually converges to it.

To summarize briefly our results so far, we have identified two conceptually different diffusion regimes. In the first regime ($\alpha_1 \geq \alpha_2$ and $\beta_2 \geq \beta_1$), every initial state transits into a universal diffusion mode given by the space-time separable solution. The decay rate of this asymptotic solution is known a priori unles $\alpha_2\beta_1 = \alpha_1\beta_2$, in which case a global analysis is needed. In the second regime, diffusion is inhibited and, although the system evolves toward the separable form, it cannot, mathematically speaking, attain this form. The nonlinear interplay between density and the temperature always inhibits the diffusion of either temperature or density.

3. Exact Integrals of the Emden-Fowler Equation

The celebrated Emden-Fowler equation (henceforth referred to as E.F.E.) appears in various branches of physics and engineering and as such was and still is a subject of extensive analysis. A review by Wong [1] summarizes the investigations concerning the qualitative properties of this equation and its generalizations. In a different vein Ames and Adams [2] employ

a group method to transform the E.F.E. stated as a boundary value problem, into an initial value problem which then becomes an easy numerical task.

Our interest is different; we focus our attention on an analytical integration of E.F.E. written as

$$xy'' + zy' + ax^m y^n = 0, \quad a = \text{const.,} \tag{1a}$$

or in one of its slightly generalized forms

$$xy'' + (1+\beta) y' + ax^m y^n = 0 \tag{1b}$$

or $\quad (x^{\nu+\alpha} y')' + ax^\nu y^n = 0 \tag{1c}$

where $\quad m = 1-\alpha, \ \beta = \nu - m.$

Related to our interest are several approaches for finding first integrals of dynamical systems that have been recently presented. They use the E.F.E., (Eq.(1a)) or one of its variants (Eqs. (1b) or (1c)) as a test case to demonstrate the applicability of the advocated method. These procedures, generally speaking, are either variational, group-variational (i.e., exploit a la Noether the invariance of the Lagrangian) or, use an invariance property of partial differential equations that the searched-after first integral satisfies. These methods will be reviewed and compared with our approach. In our work, however, we approach this problem directly and derive two sufficient conditions, of which at least one appears to be the hitherto unknown. When either of these conditions is satisfied, one is ensured not only of the existence of first integral of

motion but a total integration of E.F.E. The point that every first integral of E.F.E. may be brought to an autonomous form and thus further integrated is trivial but was surprisingly enough unnoticed in previous works. It is noteworthy that each integrable case generates a one parameter family of integrable Emden-Fowler equations.

4. New Results in the Theory of Turbulent Self-Diffusion

The work described in this and the following section is being carried out by E. Hameiri.

We consider the diffusion of a fluid element as a result of stationary and homogeneous turbulence. Thus, we know the same-time statistics of the turbulent velocity field $\underset{\sim}{u}$ $<\underset{\sim k}{u}(0)\cdot\underset{\sim \ell}{u}^{*}(0)>$ $= a_k^2 \, \delta_{k\ell}$, and would like to determine the expected deviation $X(t)$ of a fluid particle from its original position. $Y(t) \equiv d^{-1}<|\underset{\sim}{x}(t) - \underset{\sim}{x}(0)|^2>$, where k indicates a Fourier mode and d is the dimensionality of the space.

There have been many attempts to solve this problem. For example, Taylor and McNamara, (Phys. Fluids, 1973), after a number of approximations obtained the result

$$\frac{d^2}{dt^2} Y = \frac{2}{d} \sum_k e^{-1/2 k^2 Y} a_k^2 \qquad Y(0) = \overset{o}{Y}(0) = 0$$

which determines $Y(t)$. The result was below, but rather close to, numerical calculations. This work however had some short-comings. First, it dealt only with the two-dimensional case d=2. Secondly, it required some precise knowledge of the flow, namely

that the vorticity was advected by the fluid, and the approximation making use of this information appeared only one of many others that could have been made.

Our approach circumvents the need for knowledge of the flow by using a Lagrangian variational principle which determines it. Using then a well known (but not necessarily valid) approximation for a turbulent medium (based on non-dynamic considerations) as an additional constraint, we get the result

$$\frac{d^2}{dt^2} Y \geq \frac{2}{d}(\Sigma a_k^2)^{1/2} (\Sigma e^{-k^2 Y} a_k^2)^{1/2}$$

Inequality holds because of the dropping of a constraint (the knowledge of the end points of all particle trajectories in time). This result is obtained for arbitrary d, and is in striking contrast to Taylor's relation. A Schwartz inequality argument shows our Y(t) to be larger than Taylor's. Our result can be improved to give a somewhat closer bound by using additional information on the flow as further constraints on the variation, e.g., conservation of enstrophy.

This novel approach to the diffusion problem can be similarly used by employing other variational principles which determine the same flow. In particular, one can get a lower bound on the diffusion by using a principle "dual" to the first one. We did not succeed yet in obtaining a lower bound apparently close to Taylor's and further analytical and numerical work is needed.

5. Mathematical Theory of the Essential Spectrum in Magneto-hydrodynamics

The linearized MHD equations present a rare example of differential operator which has a non pure point spectrum, even though it may be defined on a finite domain and with coefficients as smooth as we wish. This stands in marked contrast to the much discussed Schrödinger operator. From a practical point of view, the MHD spectrum determines the time evolution of small perturbations of a plasma about an equilibrium state. The singularities in the spectrum appear in the configurations common in magnetic confinement experiments, but the underlying causes affect the behavior of all magnetized plasmas, e.g., solar wind plasma.

The present work represents a rigorous mathematical investigation of the subject. Some results were nevertheless known previously, usually derived by heuristic arguments. In particular, Grad, Pao and others derived equations which determined the so-called "Alfven" system, while the existence of "ballooning modes" was discovered in the past few years. These latter modes, which were derived as a stability criterion, are shown by us to be another part of the essential spectrum. (This spectrum is defined as the whole spectrum except for all discrete eigenvalues of finite multiplicity.)

The derivation of the essential spectrum is done by us by finding an "approximate eigenfunction", i.e., by constructing a sequence of functions, not converging to zero in the norm, such

that the eigenvalue equation is satisfied in the limit. It
always happens that the limiting function needs to be localized
in space near a single field line, and a reduced one-dimensional
eigenvalue equation along the field line determines points in
the essential spectrum of the original problem. There are many
possible directions of strong localization, one of which yields
the "Alfven" spectrum and others the "ballooning spectrum." In
the case of axisymmetry, the spectrum is the union of the spectra
corresponding to Fourier modes in the ignorable direction, and
we proved that each such part consists of exactly an Alfven
spectrum plus discrete modes. Thus ballooning modes must be
the accumulation points of discrete eigenvalues, the accumula-
tion occuring over non empty intervals. Our proof here utilized
the properties of compact operators in the theory of perturbations
of the spectrum of operators.

Finally, we gave a physical interpretation to the essential
spectrum. The need for localized eigenfunctions indicates the
existence of one-dimensional wave propagation along magnetic
field lines. Indeed, one such wave is known as the Alfven wave,
and in the linearized system another such wave exists. Looked
at from this point of view, solutions of the localized equations
we obtained are needed in order to determine plasma behavior
in all configurations with a magnetic field, e.g., whistler
plasmas.

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